

The Emerton Gee Stack

-Introduction

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- Mazur - formal deformation theory of Gal reps. to study infinitesimal neighborhoods of points in moduli spaces
- Kisin - suggested the existence of formal algebraic stacks over $\mathbf{Z}_p$ with closed points corresponding to residual Gal reps and whose versal rings at closed pts recover Gal def. rings
- Wang-Erickson - literal moduli stack of cts Gal rep $\rho : G_K \rightarrow GL_d(A)$ for K p -adic field and A p -adically complete $\mathbf{Z}_p$ algebra
- Emerton and Gee - Moduli stacks of etale (φ, Γ) modules that contain Wang-Erickson's stacks as substacks and are the same at the level of $\bar{\mathbf{F}}_p$ points

- P-adic comparison theorem compares p-adic etale and algebraic de-Rham coh of a variety over a local field K
- We want to capture this comparison isomorphism into an object corresponding to the variety s.t. we remember the cohomology groups
- Fontaine- theory of (φ, Γ) -modules for $\mathbf{Z}_p$ -representations and p-adic Hodge theory allows study of relationships among cohomology theories associated to alg. varieties over local fields .
- Colmez's work on the p-adic L.L.C. and Kedlaya-Liu's work- Consider families of (φ, Γ) modules rather than considering families of representations of G_K .

- (φ) and (φ, Γ) modules
- Moduli Stacks of (φ, Γ) modules $\mathcal{X}$
- $\mathcal{X}$ is an Ind-algebraic stack

- Fix a Z_p algebra R and $\varphi \in \text{End}(R)$, congruent to the (p-power) Frobenius modulo p .
- An etale φ -module over R is a finite R -module M , with a φ -semi-linear endomorphism $\varphi_M : M \rightarrow M$, s.t. $R \otimes_{R,\varphi} M \rightarrow M$ is an isomorphism

- Set $\mathbf{A}_K = W(k)((T))^{\wedge}$
- Actions by Frobenius φ and profinite group Γ commuting with φ
- An etale (φ, Γ) module M is defined to be a finite $\mathbf{A}_K$ module, with commuting semi-linear actions of φ and Γ , such that the linearized action of φ from $\mathbf{A}_K \otimes_{\mathbf{A}_{K,\varphi}} M$ to M given by $1 \otimes \varphi_M$ is an isomorphism
- For a p -adically complete Z_p -algebra A . We define etale (φ, Γ) -module with A -coefficients analogously using the ring $\mathbf{A}_{K,A} := A \hat{\otimes} \mathbf{A}_K$

- Fix $K/\mathbf{Q}_p$ unramified.
- Consider any p -adically complete $\mathbf{Z}_p$ algebra A
- We define the A points of $\mathcal{X}_{K,d}$ i.e. $\mathcal{X}_{K,d}(A)$ to be the groupoid of projective étale (φ, Γ) -modules of rank d with A -coefficients, with morphisms given by isomorphisms.

The equivalence between (φ, Γ) modules and G_K reps over finite type $\mathbf{Z}_p$ algebras gives us a bijection between the $\bar{\mathbf{F}}_p$ points of $\mathcal{X}_{K,d}$ and Galois representations $\bar{\rho} : G_K \rightarrow GL_d(\mathbf{F}_p)$

Quick recap of relevant definitions

- Ind-algebraic stack over a scheme is a stack $\mathcal{X} \cong \lim_{i \in I} \mathcal{X}_i$, where we take a 2-colimit in the 2-category of stacks of a 2 directed system $\mathcal{X}_{i \in I}$.
- A morphism of stacks $\mathcal{X} \cong \lim_{\rightarrow} \mathcal{X}_{\lambda} \rightarrow \mathcal{Y}$ is Ind-quasi-compact if each morphism $\mathcal{X}_{\lambda} \rightarrow \mathcal{Y}$ is quasi compact and is representable by algebraic stacks.
- Suppose also, $\mathcal{Y}$ is Ind-algebraic i.e. $\mathcal{Y} \cong \lim_{\rightarrow} \mathcal{Y}_{\mu}$ and assume that $\mathcal{X}_{\lambda} \rightarrow \mathcal{Y}_{\mu}$ are quasi-compact morphisms. Then, each induced map has a scheme theoretic image $\mathcal{Z}_{\lambda}$ that's a closed substack of $\mathcal{Y}$. The scheme theoretic image of the Ind-quasi compact morphism $\mathcal{X} \rightarrow \mathcal{Y}$ is $\mathcal{Z} := \lim_{\rightarrow} \mathcal{Z}_{\lambda}$.

Steps

- Step 0 Setting up Notation
- Step 1 Moduli Stacks of φ modules $\mathcal{R}_d^a$ and the 2 colimit $\mathcal{R}_d$
- Step 2 Notion of height
- Step 3 Moduli Stacks $\mathcal{C}_{d,h}^a$ and $\mathcal{C}_{d,h}$
- Step 4 Constructing $\mathcal{R}_d^{\Gamma_{disc}}$
- Step 5 Weak Wach Modules and Level of Wach Modules
- Step 6 Constructing the Moduli Stacks $\mathcal{W}_{d,h}$ and $\mathcal{W}_{d,h,s}$
- Connecting all the stacks and the proof

- Fix a finite extension k/F_p
- Write $\mathbf{A}^+ := W(k)[[T]]$ and $\mathbf{A}$ for the p-adic completion of $\mathbf{A}^+[1/T]$.
- For a p-adically complete Z_p -algebra A , write $\mathbf{A}_A^+ := (W(k) \otimes_{Z_p} A)[[T]]$
- $\mathbf{A}_A$ is the p-adic completion of $\mathbf{A}_A^+[1/T]$

Step 1 The stack $\mathcal{R}_d^a$

- Fix a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field F .
- Fix integers $a, d \geq 1$. For any $\mathcal{O}/\varpi^a$ -algebra A , $\mathcal{R}_d^a(A) :=$ groupoid of étale φ -modules over $\mathbf{A}_A$ which are projective of rank d .

Step 1 continued: $\mathcal{R}_d$

- $\mathcal{R}_d := \lim_{\rightarrow a} \mathcal{R}_d^a$ i.e. 2 co-limit letting a vary.
- $\mathcal{R}_d$ is a stack over $\mathrm{Spf} \mathcal{O}$.

Step 2 : Notion of height

- Fix $F \in W(k)[T]$ s.t. $F \cong T^n \bmod p$ for some positive n .
- Let h be a non-negative integer, and let A be a p -adically complete $\mathcal{O}$ -algebra.
- A φ -module of F -height at most h over $\mathbf{A}_A^+$ is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ satisfying the conditions below.
 - $\mathfrak{M}$ is a finitely generated T -torsion free $\mathbf{A}_A^+$ module
 - $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ a φ -semi-linear map.
 - $1 \otimes \varphi_{\mathfrak{M}} : \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is injective and its cokernel is annihilated by F^h .

Step 3 Constructing the Moduli Stacks $\mathcal{C}_{d,h}^a$ and $\mathcal{C}_{d,h}$

- $\mathcal{C}_{d,h}^a$ is defined over $\mathrm{Spec}\mathcal{O}/\varpi^a$ s.t for any $\mathcal{O}/\varpi^a$ algebra A ,
 $\mathcal{C}_{d,h}^a(A) :=$ the groupoid of φ -modules of F-height at most h over $\mathbf{A}_A^+$,
which are projective of rank d .
- $\mathcal{C}_{d,h} := \lim_{\rightarrow a} \mathcal{C}_{d,h}^a$ over $\mathrm{Spf}\mathcal{O}$

- Let $\Gamma = \Gamma_K$ and γ be its topological generator.
- $\Gamma_{disc} := \langle \gamma \rangle$
- Canonical morphism $\gamma : \mathcal{R}_d \rightarrow \mathcal{R}_d$, where for M an object of $\mathcal{R}_d(A)$, where $\gamma(M) := \mathbf{A}_{K,A} \otimes_{\gamma, \mathbf{A}_{K,A}} M$.
- $\mathcal{R}_d^{\Gamma_{disc}} := \mathcal{R}_d \times_{\Delta, R_d \times R_d, \Gamma_\gamma} \mathcal{R}_d$, where Δ is the diagonal of R_d and Γ_γ is the graph of γ

Step 5 Weak Wach Modules

- Let A be a $\mathcal{O}/\varpi^a$ -algebra for some $a \geq 1$
- A rank d projective weak Wach module of T -height $\leq h$ with A -coefficients is
 - a rank d projective φ -module M over $\mathbf{A}_{K,A}^+$
 - with T -height at most h
 - $M[1/T]$ is equipped with a semi-linear action of Γ
 - The Γ action is continuous when $M[1/T]$ is endowed with its canonical topology.

- Fix a rank d projective φ -module M of T -height $\leq h$ over A , such that $M[1/T]$ is equipped with a semi-linear action of Γ_{disc} .
- The Γ_{disc} action extends to a continuous action of Γ iff. $\exists s \geq 0$ we have $(\gamma^{p^s} - 1)\mathfrak{M} \subseteq T\mathfrak{M}$.
- In this case we say that $\mathfrak{M}$ has level $\leq s$.

Step 6 Constructing the Moduli Stacks $\mathcal{W}_{d,h}$ and $\mathcal{W}_{d,h,s}$

- $\mathcal{W}_{d,h} :=$ the moduli stack of rank d projective weak Wach modules of T -height $\leq h$
- $\mathcal{W}_{d,h,s} :=$ The substack of rank d projective weak Wach modules of T -height $\leq h$ and level $\leq s$.

- $\mathcal{W}_{d,h}$ may be treated as a substack of $\mathcal{R}_d^{disc} \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$
- Restricting the Γ -action on a weak Wach module to Γ_{disc} gives a fully faithful morphism $\mathcal{W}_{d,h} \rightarrow \mathcal{R}_d^{disc} \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$.
- We will assume that $\mathcal{R}_d^a$ is an Ind algebraic stack
- E.G. prove this by showing that $\mathcal{C}_{d,h}^a \rightarrow \mathcal{R}_d^a$ has a scheme theoretic image that's an algebraic substack and they show $\lim_{\rightarrow} \mathcal{R}_{d,h}^a \rightarrow \mathcal{R}_d^a$ is an isomorphism.

$$\begin{array}{ccc} \mathcal{W}_{d,h} & \longrightarrow & \mathcal{R}_d^{disc} \times_{\mathcal{R}_d} \mathcal{C}_{d,h} \\ \downarrow & & \downarrow \\ \mathcal{X}_d & \longrightarrow & \mathcal{R}_d^{disc} \end{array}$$

Prove $\mathcal{X}_d$ is an Ind-algebraic stack by showing it's the “scheme-theoretic image” of the morphism $\lim_{\rightarrow h} \mathcal{W}_{d,h} \rightarrow \mathcal{R}_d^{disc}$

- We choose a level $s \geq 0$ and consider $\mathcal{W}_{d,h,s} \rightarrow \mathcal{W}_{d,h} \rightarrow \mathcal{X}_d \rightarrow \mathcal{R}_d^{disc}$
- Also factorizes as $\mathcal{W}_{d,h,s} \rightarrow \mathcal{W}_{d,h} \rightarrow \mathcal{R}_d^{disc} \times_{\mathcal{R}_d} \mathcal{C}_{d,h} \rightarrow \mathcal{R}_d^{disc}$
- Define a closed algebraic substack of $\mathcal{W}_{d,h,s}$ as
 $\mathcal{W}_{d,h,s}^a := \mathcal{W}_{d,h,s} \times_{Spf \mathcal{O} / Spec \mathcal{O} / \varpi^a} \text{ for } a \in \mathbf{Z}_{\geq 1}.$
- $\mathcal{X}_{d,h,s}^a :=$ schem-theoretic image of the composite morphism
 $\mathcal{W}_{d,h,s}^a \rightarrow \mathcal{W}_{d,h,s} \rightarrow \mathcal{R}_d^{\Gamma^{disc}}$
- $\mathcal{X}_{d,h,s}^a$ is a closed substack of $\mathcal{X}_d$

The canonical morphism $\lim_{\rightarrow} \mathcal{X}_{d,h,s}^a \rightarrow \mathcal{X}_d$ is an isomorphism.

- Moduli stacks of etale (φ, Γ) modules and the existence of crystalline lifts - Matthew Emerton and Toby Gee
- On categories of (φ, Γ) modules-Kiran S. Kedlaya and Jonathan Pottharst
- Relative p-adic Hodge Theory: Foundations - Kiran S. Kedlaya and Ruochuan Liu

THANK YOU!